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On the Problem of Dynamical Localization in the Nonlinear Schrödinger Equation with a Random Potential

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Abstract We prove a dynamical localization in the nonlinear Schrödinger equation with a random potential for times of the order of $O(\beta^{-2})$, where β is the strength of the nonlinearity.

Keywords Anderson localization · NLSE · Random potential · Nonlinear Schrödinger · Dynamical localization

1 Introduction

We consider the problem of a dynamical localization of waves in a Nonlinear Schrödinger Equation (NLSE) with a random potential term:

$$i\partial_t\psi = (-\partial_{xx} + V_\omega)\psi + \beta|\psi|^2\psi, \tag{1.1}$$

where $\psi = \psi(x,t)$, $x \in \mathbb{Z}$ (or $\mathbb{R}$; here we specialize to the discrete case) and $\{V_\omega\}_{\omega\in\Omega}$ is a collection of random potentials chosen from the set Ω, with probability measure $\mu(\omega)$. We assume that localization is known for *all* the energies of the linear problem, when $\beta = 0$.

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The NLSE was derived for a variety of physical systems under some approximations. It was derived in classical optics where ψ is the electric field by expanding the index of refraction in powers of the electric field keeping only the leading nonlinear term [1]. For Bose-Einstein Condensates (BEC), the NLSE is a mean field approximation where the density $\beta|\psi|^2$ approximates the interaction between the atoms. In this field the NLSE is known as the Gross-Pitaevskii Equation (GPE) [2–7]. Recently, it was rigorously established, for a large variety of interactions and of physical conditions, that the NLSE (or the GPE) is exact in the thermodynamic limit [8, 9]. Generalized mean field theories, where several mean-fields are used, were recently developed [10, 11]. In the absence of randomness (1.1) is completely integrable. For repulsive nonlinearity ($\beta > 0$) an initially localized wavepacket spreads, while for attractive nonlinearity ($\beta < 0$) solitons are found in general [12].

It is well known that in 1D in the presence of a random potential and in the absence of nonlinearity ($\beta = 0$) with probability one all the states are exponentially localized [13–16]. Consequently, diffusion is suppressed and in particular a wavepacket that is initially localized will not spread to infinity. This is the phenomenon of Anderson localization. In 2D it is known heuristically from the scaling theory of localization [16, 17] that all the states are localized, while in higher dimensions there is a mobility edge that separates localized and extended states. This problem is relevant to experiments in nonlinear optics, for example disordered photonic lattices [18], where Anderson localization was found in presence of nonlinear effects as well as experiments on BECs in disordered optical lattices [19–25]. The interplay between disorder and nonlinear effects leads to new interesting physics [23, 24, 26–30]. In spite of the extensive research, many fundamental problems are still open, and, in particular, it is not clear whether in one dimension (1D) Anderson localization can survive the effects of nonlinearities. This will be studied here.

A natural question is whether a wave packet that is initially localized in space will indefinitely spread for dynamics controlled by (1.1). A simple argument indicates that spreading will be suppressed by randomness. If unlimited spreading takes place the amplitude of the wave function will decay since the L^2 norm is conserved. Consequently, the nonlinear term will become negligible and Anderson localization will take place as result of randomness. A different argument for localization is given in [31]. Contrary to these arguments, it is claimed that for the kicked-rotor and the Anderson model a nonlinear term leads to delocalization if it is strong enough [32, 33]. It is predicted in that work that there is a critical value of β that separates the occurrence of localized and extended states. In the delocalized regime sub-diffusion is found. In a different work [34] sub-diffusion was reported for all values of β, but with a different power of the time dependence (compared with [32]). The work of [31], presents results of extensive numerical calculation and heuristic arguments that shed light on this problem. It was also argued that nonlinearity may enhance discrete breathers [29]. In conclusion, it is *not* clear what is the long time behavior of a wave packet that is initially localized, if both nonlinearity and disorder are present. This is the main motivation for the present work. Since heuristic arguments and numerical simulations produce conflicting results, rigorous statements are required for further progress.

More precisely, the question of dynamical localization can be rigorously formulated as follows: assume the initial state is $\psi(x,0) \equiv u_0(x)$, $x \in \mathbb{Z}$; for any $0 < \varepsilon < 1$, prove that with probability $1 - \varepsilon$ (on the space of the potentials)

$$\sup_{x,t} |e^{a|x|}\psi(x,t)| < M_\varepsilon < \infty \tag{1.2}$$

for some $a > 0$.

Rigorous results on dynamical localization for the linear case are well known [35–39]. However, the nonlinear problem turns out to be very difficult to handle, even numerically. Consider the case of small β (the only case we study in this work). There are two possible mechanisms for destruction of the localization due to nonlinearity.

One way of spreading is to spread into many random places with increasing number of them. In that case, due to conservation of the normalization of the solution, the solution becomes small (for simplicity, assume $x \in \mathbb{Z}$. If $x \in \mathbb{R}$, we need to have uniform continuity as well). But then, the nonlinear term becomes less and less important and we expect the linear theory to take over and lead to localization. While this argument sounds plausible there is no proof along this lines.

The second way of spreading is in a few fixed number of spikes that hop randomly to infinity. In this case, the nonlinear term is always relevant. It is this (possible) process that makes the proof of localization in the nonlinear case so elusive. It also precludes a quick numerical analysis of the problem: it may take exponentially long time to see the hoping.

Rigorous results in this direction are of preliminary nature: In [40] it was shown that dynamical localization holds for the linear problem perturbed by a periodic in time and exponentially localized in space small linear perturbation. In [41] the above result was extended to a quasiperiodic in time perturbation. Such perturbations mimic the nonlinear term:

$$|\psi|^2 \to \left|\sum_j c_j u_j(x) e^{iE_j t}\right|^2, \tag{1.3}$$

where u_i are the eigenfunctions of the linear problem with energies E_j.

Furthermore, it can be shown that NLSE has stationary solutions

$$E\psi_E = (-\partial_{xx} + V_\omega)\psi_E + \beta|\psi_E|^2\psi_E \tag{1.4}$$

which are exponentially localized for almost all E with a localization length that is identical to the one of the linear problem [42–45].

In this work we prove that for times of order $O(\beta^{-2})$ the solution remains exponentially localized. The argument we use seems to generalize to arbitrary order in β^{-1} as will be explained later. We construct the solution as a series in the eigenfunctions of the linear problem. Standard perturbation theory for the coefficients does not apply: we encounter small divisor problems and secular terms (formally infinite).

Removing the secular terms requires the renormalization of the original linear Hamiltonian that is used to generate the expansion through its eigenfunctions. Small divisor terms are estimated by inspiration from the work and methods of Aizenman-Molchanov [46]. The nonlinear terms which are then controlled by a bootstrap argument, utilizing at this point the smallness of β.

We use only the fact that the eigenstates of the linear problem are localized, therefore our results hold in many situations for dimensions higher than one in the presence of strong disorder.

2 Organization of the Perturbation Theory

2.1 The Perturbation Expansion

Our goal is to analyze the nonlinear Schrödinger equation

$$i\partial_t\psi = H_0\psi + \beta|\psi|^2\psi, \tag{2.1}$$

where H_0 is the linear part with a disordered potential, which on the lattice takes the form of

$$H_0\psi_n = -J(\psi_{n+1} + \psi_{n-1}) + \varepsilon_n\psi_n. \tag{2.2}$$

We assume throughout the paper that H_0 satisfies the conditions for localization and the conditions of [46]. For almost all the realizations, ω, of the disordered potential, all the eigenstates of H_0, u_m, are exponentially localized and have an envelope of the form of

$$|u_m(x)| \le D_{\omega,\varepsilon} e^{\varepsilon|x_m|} e^{-\gamma|x-x_m|}, \tag{2.3}$$

where $\varepsilon > 0$, x_m is the localization center defined by the point where $u_m(x)$ has its maximum, γ is the inverse of the localization length, $\xi = \gamma^{-1}$, and $D_{\omega,\varepsilon}$ is a constant dependent on ε and the realization of the disordered potential [50, 51] (better estimates were proven recently in [52, 53]). It is of importance that $D_{\omega,\varepsilon}$ does not depend on the energy of the state. In the present work, only realizations ω, where

$$|D_{\omega,\varepsilon}| \le D < \infty \tag{2.4}$$

are considered. This is satisfied for a set of order $1-\varepsilon$, as can be easily verified by a contradiction argument.

We exclude this way also the realizations where functions with more than one center of localization are found at large distances. This is the reason that our results hold for a set of measure $1-\varepsilon$ of the potentials.

Our goal is to introduce an indexing scheme which follows the localization center of the eigenstates, u_m. For each point $i \in \mathbb{Z}$ on the lattice, we label the eigenfunctions with the localization center at i, and ordered by energy. Of course, as we move from i to $i+1$ we do not recount the eigenfunctions already present in the previous count. We start with $i = 0$ and go back and forward in i to $\pm\infty$. The crucial fact to us is that the number of eigenfunctions for each i point is uniformly bounded in i. Consequently, if the eigenfunctions index is n, we have that

$$c|i| \le |n| \le c'|i| \tag{2.5}$$

for some $0 < c < c' < \infty$. Numerically one observes that $c \sim c' \sim 1$, hence the localization center, x_n, of the nth eigenfunction is of the order n

$$n \lesssim x_n \lesssim n. \tag{2.6}$$

The fact that the number of eigenfunctions localized at any point (say $i = 0$) is finite follows from the results of Nakano [54]. Another way to see that is to note that it follows from a complete localization. Complete localization implies that in a box of size M around the origin all eigenfunctions with $x_n = 0$ satisfy

$$\sum_{|j|<M} |u_n(j)|^2 \ge 1 - \varepsilon(M) \tag{2.7}$$

with $\varepsilon(M) = O(e^{-M})$. The dimension of the space of functions which are completely localized inside M is M, spanned by $u_1, \dots, u_M$. So, suppose there is another independent eigenfunction ψ satisfying

$$\sum_{|j|<M} |\psi(j)|^2 \ge 1 - \varepsilon(M), \quad \langle\psi|u_j\rangle = 0, \quad j = 1, \dots, M. \tag{2.8}$$

Then

$$\sum_{j=1}^{M}\{\langle\chi_M\psi|u_j\rangle+\langle(1-\chi_M)\psi|u_j\rangle\}=0, \tag{2.9}$$

where χ_M is the projection operator on the box M. But $\{\chi_M u_j\}$ span the space of functions inside M, hence

$$\left|\sum_{j=1}^{M}\langle\chi_M\psi|u_j\rangle\right|\geq(1-\varepsilon(M))(1-\varepsilon) \tag{2.10}$$

for ε small. Therefore (2.9) implies

$$\left|\sum_{j=1}^{M}\langle(1-\chi_M)\psi|u_j\rangle\right|\geq(1-\varepsilon(M))(1-\varepsilon). \tag{2.11}$$

But by the construction of the box

$$\left|\sum_{j=1}^{M}\langle(1-\chi_M)\psi|u_j\rangle\right|\leq cMe^{-\gamma M}\ll(1-\varepsilon(M)) \tag{2.12}$$

for M large results in a contradiction.

The wavefunctions can than be expanded using the eigenstates of H_0 as

$$\psi=\sum_m c_m e^{-iE_m t}u_m(x). \tag{2.13}$$

For the nonlinear equation the dependence of the expansion coefficient, $c_m(t)$, is found by inserting this expansion into (2.1), resulting in

$$i\partial_t\sum_m c_m e^{-iE_m t}u_m(x)=H_0\sum_m c_m e^{-iE_m t}u_m(x)$$
$$+\beta\left|\sum_m c_m e^{-iE_m t}u_m(x)\right|^2\sum_{m_3}c_{m_3}e^{-iE_{m_3}t}u_{m_3}(x). \tag{2.14}$$

Multiplying by $u_n(x)$ and integrating gives:

$$i\partial_t c_n=\beta\sum_{m_1,m_2,m_3}V_n^{m_1m_2m_3}c_{m_1}^*c_{m_2}c_{m_3}e^{i(E_{m_1}+E_n-E_{m_2}-E_{m_3})t}, \tag{2.15}$$

where $V_n^{m_1m_2m_3}$ is an overlap integral

$$V_n^{m_1m_2m_3}=\int dx\cdot u_n(x)u_{m_1}(x)u_{m_2}(x)u_{m_3}(x). \tag{2.16}$$

In the discrete case $\int dx$ is understood as $\sum_x$. By definition $V_n^{m_1m_2m_3}$ is symmetric with respect to interchange of any two indices. Additionally, since the $u_n(x)$ are exponentially localized around x_n, $V_n^{m_1m_2m_3}$ is not negligible only when the interval,

$$\delta m\equiv\max[x_n,x_{m_i}]-\min[x_n,x_{m_i}], \tag{2.17}$$

is of the order of the localization length, around x_n,

$$
\begin{aligned}
|V_n^{m_1m_2m_3}| &\le D^4_{\omega,\varepsilon} e^{\varepsilon(|x_n|+|x_{m_1}|+|x_{m_2}|+|x_{m_3}|)} \int dx \cdot e^{-\gamma(|x-x_n|+|x-x_{m_1}|+|x-x_{m_2}|+|x-x_{m_3}|)} \\
&\le D^4_{\omega,\varepsilon} e^{\varepsilon(|x_n|+|x_{m_1}|+|x_{m_2}|+|x_{m_3}|)} e^{-\frac{(\gamma-\varepsilon')}{3}(|x_n-x_{m_1}|+|x_n-x_{m_2}|+|x_n-x_{m_3}|)} \\
&\quad \times \int dx \cdot e^{-\varepsilon'(|x-x_n|+|x-x_{m_1}|+|x-x_{m_2}|+|x-x_{m_3}|)} \\
&\le V_\omega^{\varepsilon,\varepsilon'} e^{\varepsilon(|x_n|+|x_{m_1}|+|x_{m_2}|+|x_{m_3}|)} e^{-\frac{1}{3}(\gamma-\varepsilon')(|x_n-x_{m_1}|+|x_n-x_{m_2}|+|x_n-x_{m_3}|)}. \qquad (2.18)
\end{aligned}
$$

Here we used the triangle inequality

$$
\begin{aligned}
&(|x-x_n|+|x-x_{m_1}|)+(|x-x_n|+|x-x_{m_2}|)+(|x-x_n|+|x-x_{m_3}|) \\
&\quad \ge |x_n-x_{m_1}|+|x_n-x_{m_2}|+|x_n-x_{m_3}| \qquad (2.19)
\end{aligned}
$$

to obtain the second line. Our objective is to develop a perturbation expansion of the $c_m(t)$ in powers of β and to calculate them order by order in β. The resulting expansion is

$$
c_n(t) = c_n^{(0)} + \beta c_n^{(1)} + \beta^2 c_n^{(2)} + \cdots + \beta^{N-1} c_n^{(N-1)} + \beta^N c_n^{(r)}, \qquad (2.20)
$$

where the expansion is till order $(N-1)$ and $c_n^{(r)}$ is the remainder of the expansion. We will assume the initial condition

$$
c_n(t=0) = \delta_{n0}. \qquad (2.21)
$$

The equations for the two leading orders are presented in what follows.

The leading order is

$$
c_n^{(0)} = \delta_{n0}. \qquad (2.22)
$$

The equation for the first order is

$$
\partial_t c_n^{(1)} = -i \sum_{m_1,m_2,m_3} V_n^{m_1m_2m_3} c_{m_1}^{*(0)} c_{m_2}^{(0)} c_{m_3}^{(0)} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t} = -i V_n^{000} e^{i(E_n-E_0)t} \qquad (2.23)
$$

and its solution is

$$
c_n^{(1)} = V_n^{000}\left(\frac{1-e^{i(E_n-E_0)t}}{E_n-E_0}\right). \qquad (2.24)
$$

The resulting equation for the second order is

$$
\begin{aligned}
\partial_t c_n^{(2)} &= -i \sum_{m_1,m_2,m_3} V_n^{m_1m_2m_3} c_{m_1}^{*(1)} c_{m_2}^{(0)} c_{m_3}^{(0)} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t} \\
&\quad - 2i \sum_{m_1,m_2,m_3} V_n^{m_1m_2m_3} c_{m_1}^{*(0)} c_{m_2}^{(1)} c_{m_3}^{(0)} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}. \qquad (2.25)
\end{aligned}
$$

Substitution of the lower orders yields

$$
\partial_t c_n^{(2)} = -i \sum_m V_n^{m00} V_m^{000} \left[\left(\frac{1-e^{-i(E_m-E_0)t}}{E_m-E_0}\right) e^{i(E_n+E_m-2E_0)t}\right.
$$

$$+2\left(\frac{1-e^{i(E_m-E_0)t}}{E_m-E_0}\right)e^{i(E_n-E_m)t}\Bigg]$$

$$=-i\sum_m \frac{V_n^{m00}V_m^{000}}{E_m-E_0}[(e^{i(E_n+E_m-2E_0)t}-e^{i(E_n-E_0)t})+2(e^{i(E_n-E_m)t}-e^{i(E_n-E_0)t})]$$

$$=-i\sum_m \frac{V_n^{m00}V_m^{000}}{E_m-E_0}[e^{i(E_n+E_m-2E_0)t}-3e^{i(E_n-E_0)t}+2e^{i(E_n-E_m)t}]. \tag{2.26}$$

We notice that divergence of this expansion may result from three major problems: the secular terms problem, the entropy problem (i.e., factorial proliferation of terms), and the small denominators problem. In the present work, the entropy problem is irrelevant since only the lowest order is studied.

2.2 Elimination of Secular Terms

We first show how to derive the equations for $c_n(t)$ where the secular terms are eliminated.

Proposition 2.1 *To each order in* β, $\psi(x,t)$ *can be expanded as*

$$\psi(x,t)=\sum_n c_n(t)e^{-iE'_n t}u_n(x) \tag{2.27}$$

with

$$E'_n\equiv E_n^{(0)}+\beta E_n^{(1)}+\beta^2 E_n^{(2)}+\cdots \tag{2.28}$$

and $E_n^{(0)}$ *are the eigenvalues of* H_0, *in such a way that there are no secular terms to any given order.*

Here we first develop the general scheme for the elimination of the secular terms and then demonstrate the construction of E'_n when the $c_n(t)$ are calculated to the second order in β (see (2.44, 2.45)).

Inserting the expansion into (2.1) yields

$$i\sum_m[\partial_t c_m-iE'_m c_m]e^{-iE'_m t}u_m(x)$$

$$=\sum_m E_m^{(0)}c_m e^{-iE'_m t}u_m(x)$$

$$+\beta\sum_{m_1m_2m_3} c^*_{m_1}c_{m_2}c_{m_3}e^{i(E'_{m_1}-E'_{m_2}-E'_{m_3})t}u_{m_1}(x)u_{m_2}(x)u_{m_3}(x). \tag{2.29}$$

Multiplication by $u_n(x)$ and integration gives

$$i\partial_t c_n=(E_n^{(0)}-E'_n)c_n+\beta\sum_{m_1m_2m_3}V_n^{m_1m_2m_3}c^*_{m_1}c_{m_2}c_{m_3}e^{i(E'_n+E'_{m_1}-E'_{m_2}-E'_{m_3})t}, \tag{2.30}$$

where the $V_n^{m_1m_2m_3}$ are given by (2.16). Following (2.20) we expand c_n in orders of β, namely,

$$c_n=c_n^{(0)}+\beta c_n^{(1)}+\beta^2 c_n^{(2)}+\cdots, \tag{2.31}$$

resulting in the following equation for the Nth order

$$i\partial_t c_n^{(N)} = -\sum_{s=0}^{N-1} E_n^{(N-s)} c_n^{(s)} + \sum_{m_1 m_2 m_3} V_n^{m_1 m_2 m_3} \left[\sum_{r=0}^{N-1} \sum_{s=0}^{N-1-r} \sum_{l=0}^{N-1-r-s} c_{m_1}^{(r)*} c_{m_2}^{(s)} c_{m_3}^{(l)} \right] \times e^{i(E'_n + E'_{m_1} - E'_{m_2} - E'_{m_3})t}. \tag{2.32}$$

This equation gives each order in terms of the lower ones, with the initial condition of $c_n^{(0)}(t) = \delta_{n0}$. It is important to notice, that indeed for all the $c_n^{(r)}$ on the RHS, $r < N$. Secular terms are created when there are time independent terms in the RHS of the equation above. We eliminate those terms by using the first two terms in the first summation on the RHS. We make use of the fact that $c_n^{(0)} = \delta_{n0}$ and $c_n^{(1)}$ can be easily determined (see (2.35, 2.38)), and used to calculate $E_{n=0}^{(N)}$ and $E_{n\neq 0}^{(N-1)}$ that eliminate the secular terms in the equation for $c_n^{(N)}$, that is

$$E_n^{(N)} c_n^{(0)} + E_n^{(N-1)} c_n^{(1)} = E_n^{(N)} \delta_{n0} + E_n^{(N-1)} (1 - \delta_{n0}) \frac{V_n^{000}}{(E'_n - E'_0)}, \tag{2.33}$$

where only the time-independent part of $c_n^{(1)}$ was used. In other words, we choose $E_n^{(N)}$ and $E_{n\neq 0}^{(N-1)}$ so that the time-independent terms on the RHS of (2.32) are eliminated. $E_0^{(N)}$ will eliminate all secular terms with $n = 0$, and $E_n^{(N-1)}$ will eliminate all secular terms with $n \neq 0$. In the following, we will demonstrate this procedure for the first two orders, and calculate $c_n^{(1)}$.

In the first order of approximation in β we obtain

$$\begin{aligned} i\partial_t c_n^{(1)} &= -E_n^{(1)} c_n^{(0)} + \sum_{m_1 m_2 m_3} V_n^{m_1 m_2 m_3} c_{m_1}^{*(0)} c_{m_2}^{(0)} c_{m_3}^{(0)} e^{i(E'_n + E'_{m_1} - E'_{m_2} - E'_{m_3})t} \\ &= -E_n^{(1)} \delta_{n0} + V_n^{000} e^{i(E'_n - E'_0)t}. \end{aligned} \tag{2.34}$$

For $n = 0$ the equation produces a secular term

$$\begin{aligned} i\partial_t c_0^{(1)} &= -E_0^{(1)} + V_0^{000} \\ c_0^{(1)} &= it \cdot (E_0^{(1)} - V_0^{000}). \end{aligned} \tag{2.35}$$

Setting

$$E_0^{(1)} = V_0^{000} \tag{2.36}$$

will eliminate this secular term and gives

$$c_0^{(1)} = 0. \tag{2.37}$$

For $n \neq 0$ there are no secular terms in this order, therefore finally

$$c_n^{(1)} = (1 - \delta_{n0}) V_n^{000} \left(\frac{1 - e^{i(E'_n - E'_0)t}}{(E'_n - E'_0)} \right). \tag{2.38}$$

In the second order approximation in β we have

$$i\partial_t c_n^{(2)} = -E_n^{(1)} c_n^{(1)} - E_n^{(2)} c_n^{(0)} + \sum_{m_1 m_2 m_3} V_n^{m_1 m_2 m_3} (c_{m_1}^{*(1)} c_{m_2}^{(0)} c_{m_3}^{(0)} + 2 c_{m_1}^{*(0)} c_{m_2}^{(1)} c_{m_3}^{(0)})$$

$$\times\, e^{i(E'_n+E'_{m_1}-E'_{m_2}-E'_{m_3})t}$$

$$= -E_n^{(2)}\delta_{n0} - E_n^{(1)}c_n^{(1)} + \sum_{m_1} V_n^{m_1 00}(c_{m_1}^{*(1)} e^{i(E'_n+E'_{m_1}-2E'_0)t} + 2c_{m_1}^{(1)} e^{i(E'_n-E'_{m_1})t}). \quad (2.39)$$

For $n=0$ it takes the form

$$i\partial_t c_0^{(2)} = -E_0^{(2)} + \sum_{m_1} V_0^{m_1 00}(c_{m_1}^{*(1)} e^{i(E'_{m_1}-E'_0)t} + 2c_{m_1}^{(1)} e^{i(E'_0-E'_{m_1})t}).$$

Substitution of (2.35) yields

$$\begin{aligned} i\partial_t c_0^{(2)} &= -E_0^{(2)} + \sum_{m_1\neq 0} \frac{V_0^{m_1 00} V_{m_1}^{000}}{E'_{m_1}-E'_0}[(1-e^{-i(E'_{m_1}-E'_0)t})e^{i(E'_{m_1}-E'_0)t} \\ &\quad + 2(1-e^{i(E'_{m_1}-E'_0)t})e^{i(E'_0-E'_{m_1})t}] \\ &= -E_0^{(2)} + \sum_{m_1\neq 0} \frac{V_0^{m_1 00} V_{m_1}^{000}}{E'_{m_1}-E'_0}(e^{i(E'_{m_1}-E'_0)t} + 2e^{i(E'_0-E'_{m_1})t} - 3), \end{aligned} \quad (2.40)$$

and the secular term could be removed by setting

$$E_0^{(2)} = -3\sum_{m_1\neq 0} \frac{V_0^{m_1 00} V_{m_1}^{000}}{E'_{m_1}-E'_0}. \quad (2.41)$$

For $n\neq 0$ we have

$$\begin{aligned} i\partial_t c_n^{(2)} &= -E_n^{(1)} V_n^{000}\left(\frac{1-e^{i(E'_n-E'_0)t}}{(E'_n-E'_0)}\right) \\ &\quad + \sum_{m_1} V_n^{m_1 00}(c_{m_1}^{*(1)} e^{i(E'_n+E'_{m_1}-2E'_0)t} + 2c_{m_1}^{(1)} e^{i(E'_n-E'_{m_1})t}) \\ &= -E_n^{(1)} V_n^{000}\left(\frac{1-e^{i(E'_n-E'_0)t}}{(E'_n-E'_0)}\right) \\ &\quad + \sum_{m_1\neq 0} \frac{V_n^{m_1 00} V_{m_1}^{000}}{E'_{m_1}-E'_0}(e^{i(E'_n+E'_{m_1}-2E'_0)t} + 2e^{i(E'_n-E'_{m_1})t} - 3e^{i(E'_n-E'_0)t}). \end{aligned} \quad (2.42)$$

We notice that the second term in the sum produces secular terms for $m_1 = n$. Those terms could be removed by setting

$$\begin{aligned} -\frac{E_n^{(1)} V_n^{000}}{(E'_n-E'_0)} + \frac{2V_n^{n00} V_n^{000}}{E'_n-E'_0} &= 0, \quad n\neq 0 \\ E_n^{(1)} &= 2V_n^{n00}, \quad n\neq 0. \end{aligned} \quad (2.43)$$

To conclude, in the calculation of c_n up to the second order in β the perturbed energies, required to remove the secular terms, are

$$E'_n = E_n^{(0)} + \beta V_n^{n00}(2 - \delta_{n0}) - 3\beta^2 \delta_{n0} \sum_{m_1 \neq 0} \frac{(V_{m_1}^{000})^2}{E'_{m_1} - E'_0}, \tag{2.44}$$

and the corresponding correction to $c_n^{(0)}$ is

$$i\partial_t c_n^{(2)} = \begin{cases} \sum_{m_1 \neq 0} \frac{V_0^{m_1 00} V_{m_1}^{000}}{E'_{m_1} - E'_0} (e^{i(E'_{m_1} - E'_0)t} + 2e^{i(E'_0 - E'_{m_1})t}), & n = 0, \\ \frac{2V_n^{n00} V_n^{000}}{E'_n - E'_0} e^{i(E'_n - E'_0)t} + \sum_{m_1 \neq 0, n} \frac{2V_n^{m_1 00} V_{m_1}^{000}}{E'_{m_1} - E'_0} e^{i(E'_n - E'_{m_1})t} & \\ \quad + \sum_{m_1 \neq 0} \frac{V_n^{m_1 00} V_{m_1}^{000}}{E'_{m_1} - E'_0} (e^{i(E'_n + E'_{m_1} - 2E'_0)t} - 3e^{i(E'_n - E'_0)t}), & n \neq 0. \end{cases} \tag{2.45}$$

Note that in the calculation of c_n to higher orders in β, a term of the order β^2 will be generated for $n \neq 0$. Terms with increasing complexity are generated in the cancellation of higher orders, however, as demonstrated by (2.33), secular terms are removed with the same $c_n^{(0)}$ and $c_n^{(1)}$ which are presented in (2.22, 2.38).

In the next section, the simple case where the perturbation expansion is truncated at the zeroth order will be studied; and it will be shown that for short times of order β^{-2} the remainder of the perturbation expansion is bounded. Moreover, the remainder is of the order of $e^{-\gamma|n|}$.

3 Bounding the Expansion

To prove dynamical localization on the time interval of order $O(\beta^{-2})$, we need to show that for all $t \leq T = O(\beta^{-2})$ the $c_n(t)$ satisfy the following bound

$$\sup_n \sup_{t \leq T} |c_n(t) e^{a|x_n|}| < C < \infty. \tag{3.1}$$

In fact, one expects the a to be close to the inverse localization length of the linear problem. In the preceding section the terms of the perturbation expansion were examined. We now have to show that the remainder term, $c_n^{(r)}$, can be controlled. Since it is a difficult task for arbitrary N and t, at the first stage, we will obtain a bound on the remainder, $c_n^{(r)}$, for the order zero solution, $c_n^{(0)} = \delta_{n0}$. In particular, we will show that it remains exponentially bounded in n, namely,

$$|c_n^{(r)}| < M e^{-(\gamma - \varepsilon)|x_n|} \tag{3.2}$$

at least for $t < O(\beta^{-2})$. Therefore in this section we will limit ourselves to the expansion of order β, namely,

$$c_n = \delta_{n0} + \beta c_n^{(r)}, \qquad E'_n = E_n + \beta E_n^{(r)}, \tag{3.3}$$

where $c_n^{(r)}$, $E_n^{(r)}$ are the remainders of the expansions (2.20) and (2.28).

In other words, we start from an eigenstate of the linear problem ($\beta = 0$) which is localized near $x_n = 0$ and try to bound the correction $c_n^{(r)}$, resulting from the nonlinearity of the system.

The equation for $c_n^{(r)}$ is

$$i\partial_t c_n^{(r)} = -E_0^{(r)} \delta_{n0} - \beta E_n^{(r)} c_n^{(r)} + V_n^{000} e^{i(E'_n - E'_0)t}$$

$$+\beta\sum_{m_1} V_n^{m_1 00}(c_{m_1}^{(r)*}e^{i(E_n'+E_{m_1}'-2E_0')t} + 2c_{m_1}^{(r)}e^{i(E_n'-E_{m_1}')t})$$
$$+\beta^2\sum_{m_1,m_2} V_n^{m_1 m_2 0}(2c_{m_1}^{(r)*}c_{m_2}^{(r)}e^{i(E_n'+E_{m_1}'-E_{m_2}'-E_0')t} + c_{m_1}^{(r)}c_{m_2}^{(r)}e^{i(E_n'+E_0'-E_{m_1}'-E_{m_2}')t})$$
$$+\beta^3\sum_{m_1,m_2,m_3} V_n^{m_1 m_2 m_3}c_{m_1}^{(r)*}c_{m_2}^{(r)}c_{m_3}^{(r)}e^{i(E_n'+E_{m_1}'-E_{m_2}'-E_{m_3}')t}. \tag{3.4}$$

3.1 A Bound of the Linear Part

In this section we bound the linear part of the equation for $\beta^2 t < O(1)$, and in order to achieve this we integrate by parts, keeping terms of order $O(\beta^2)$. In the following section it will be shown that the nonlinear terms of order β^2 and β^3 (3.4) do not affect this result. For $n \neq 0$ the linear part is

$$\begin{aligned} i\partial_t c_n^{(r)} &= -\beta E_n^{(r)}c_n^{(r)} + V_n^{000}e^{i(E_n'-E_0')t} \\ &\quad + \beta\sum_{m_1} V_n^{m_1 00}(c_{m_1}^{(r)*}e^{i(E_n'+E_{m_1}'-2E_0')t} + 2c_{m_1}^{(r)}e^{i(E_n'-E_{m_1}')t}) \\ &= V_n^{000}e^{i(E_n'-E_0')t} + \beta\sum_{m_1} V_n^{m_1 00}c_{m_1}^{(r)*}e^{i(E_n'+E_{m_1}'-2E_0')t} \\ &\quad + 2\beta\sum_{m_1\neq n} V_n^{m_1 00}c_{m_1}^{(r)}e^{i(E_n'-E_{m_1}')t}. \end{aligned} \tag{3.5}$$

We set $E_n^{(r)} = 2V_n^{n00}$ to eliminate the secular term with $m_1 = n$. Then the terms depending on $c_n^{(r)}$ and $c_n^{(r)*}$ on the RHS of the above equation could be integrated by parts. To simplify the calculation we denote

$$\begin{aligned} I_1 &= -i\beta\int_0^t ds\sum_{m_1} V_n^{m_1 00}c_{m_1}^{(r)*}e^{i(E_n'+E_{m_1}'-2E_0')s}, \\ I_2 &= -i\beta\int_0^t ds\sum_{m_1\neq n} 2V_n^{m_1 00}c_{m_1}^{(r)}e^{i(E_n'-E_{m_1}')s}. \end{aligned} \tag{3.6}$$

So that

$$c_n^{(r)}(t) = I_1 + I_2 - i\int_0^t V_n^{000}e^{i(E_n'-E_0')t'}dt'. \tag{3.7}$$

Integrating by parts I_1 and substituting (3.5), assuming $c_m^{(r)}(t=0) = 0$, one finds

$$\begin{aligned} I_1 &= -\beta\sum_{m_1}\frac{V_n^{m_1 00}c_{m_1}^{(r)*}e^{i(E_n'+E_{m_1}'-2E_0')t}}{(E_n'+E_{m_1}'-2E_0')} \\ &\quad + i\beta\int_0^t ds\sum_{m_1}\frac{V_n^{m_1 00}}{(E_n'+E_{m_1}'-2E_0')}e^{i(E_n'+E_{m_1}'-2E_0')s}(-i\partial_t c_{m_1}^{(r)*}). \end{aligned} \tag{3.8}$$

Substitution of (3.5) leads to

$$
\begin{aligned}
I_1 = K_1 &+ i\beta \int_0^t ds \sum_{m_1} \frac{V_n^{m_1 00}}{(E'_n + E'_{m_1} - 2E'_0)} e^{i(E'_n + E'_{m_1} - 2E'_0)s} V_{m_1}^{000} e^{-i(E'_{m_1} - E'_0)s} \\
&+ i\beta^2 \int_0^t ds \sum_{m_1} \frac{V_n^{m_1 00}}{(E'_n + E'_{m_1} - 2E'_0)} e^{i(E'_n + E'_{m_1} - 2E'_0)s} \\
&\times \left(\sum_{m_2} V_{m_1}^{m_2 00} c_{m_2}^{(r)} e^{-i(E'_{m_1} + E'_{m_2} - 2E'_0)s} \right) \\
&+ i\beta^2 \int_0^t ds \sum_{m_1} \frac{V_n^{m_1 00}}{(E'_n + E'_{m_1} - 2E'_0)} e^{i(E'_n + E'_{m_1} - 2E'_0)s} \\
&\times \left(\sum_{m_2 \neq m_1} 2V_{m_1}^{m_2 00} c_{m_2}^{(r)*} e^{-i(E'_{m_1} - E'_{m_2})s} \right),
\end{aligned} \tag{3.9}
$$

where

$$
K_1 = -\beta \sum_{m_1} \frac{V_n^{m_1 00} c_{m_1}^{(r)*} e^{i(E'_n + E'_{m_1} - 2E'_0)t}}{(E'_n + E'_{m_1} - 2E'_0)}. \tag{3.10}
$$

After some manipulations it reduces to

$$
\begin{aligned}
I_1 = K_1 &+ \beta \left(\frac{e^{i(E'_n - E'_0)t} - 1}{(E'_n - E'_0)} \right) \sum_{m_1} \frac{V_n^{m_1 00} V_{m_1}^{000}}{(E'_n + E'_{m_1} - 2E'_0)} \\
&+ i\beta^2 \int_0^t ds \sum_{m_1, m_2} \frac{V_n^{m_1 00} V_{m_1}^{m_2 00} c_{m_2}^{(r)}}{(E'_n + E'_{m_1} - 2E'_0)} e^{i(E'_n - E'_{m_2})s} \\
&+ i\beta^2 \int_0^t ds \sum_{m_1} \sum_{m_2 \neq m_1} \frac{2V_n^{m_1 00} V_{m_1}^{m_2 00} c_{m_2}^{(r)*}}{(E'_n + E'_{m_1} - 2E'_0)} e^{i(E'_n + E'_{m_2} - 2E'_0)s}.
\end{aligned} \tag{3.11}
$$

Similarly for I_2,

$$
\begin{aligned}
I_2 &= -\beta \sum_{m_1 \neq n} \frac{2V_n^{m_1 00} c_{m_1}^{(r)} e^{i(E'_n - E'_{m_1})t}}{(E'_n - E'_{m_1})} - i\beta \int_0^t ds \sum_{m_1 \neq n} \frac{2V_n^{m_1 00}}{(E'_n - E'_{m_1})} e^{i(E'_n - E'_{m_1})s} (i\partial_t c_{m_1}^{(r)}) \\
&= K_2 - i\beta \int_0^t ds \sum_{m_1 \neq n} \frac{2V_n^{m_1 00}}{(E'_n - E'_{m_1})} e^{i(E'_n - E'_{m_1})s} V_{m_1}^{000} e^{i(E'_{m_1} - E'_0)s} \\
&\quad - i\beta \int_0^t ds \sum_{m_1 \neq n} \frac{2V_n^{m_1 00}}{(E'_n - E'_{m_1})} e^{i(E'_n - E'_{m_1})s} \left(\sum_{m_2} V_{m_1}^{m_2 00} c_{m_2}^{(r)*} e^{i(E'_{m_1} + E'_{m_2} - 2E'_0)s} \right) \\
&\quad - i\beta \int_0^t ds \sum_{m_1 \neq n} \frac{2V_n^{m_1 00}}{(E'_n - E'_{m_1})} e^{i(E'_n - E'_{m_1})s} \left(\sum_{m_2 \neq m_1} 2V_{m_1}^{m_2 00} c_{m_2}^{(r)} e^{i(E'_{m_1} - E'_{m_2})s} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= K_2 - \beta\left(\frac{e^{i(E'_n-E'_0)t}-1}{(E'_n-E'_0)}\right)\sum_{m_1\neq n}\frac{2V_n^{m_100}V_{m_1}^{000}}{(E'_n-E'_{m_1})}\\
&\quad - i\beta^2\int_0^t ds\sum_{m_1\neq n}\sum_{m_2}\frac{2V_n^{m_100}V_{m_1}^{m_200}c_{m_2}^{(r)*}}{(E'_n-E'_{m_1})}e^{i(E'_n+E'_{m_2}-2E'_0)s}\\
&\quad - i\beta^2\int_0^t ds\sum_{m_1\neq n,m_2\neq m_1}\frac{4V_n^{m_100}V_{m_1}^{m_200}c_{m_2}^{(r)}}{(E'_n-E'_{m_1})}e^{i(E'_n-E'_{m_2})s}
\end{aligned}
\tag{3.12}
$$

with

$$
K_2 = -\beta\sum_{m_1\neq n}\frac{2V_n^{m_100}c_{m_1}^{(r)}e^{i(E'_n-E'_{m_1})t}}{(E'_n-E'_{m_1})}. \tag{3.13}
$$

Next, we have:

Lemma 3.1 *Assume $|c_n^{(r)}(t)|e^{a|x_n|}\leq M$, uniformly in n, for all $t\leq T$. Then*

$$
\begin{aligned}
I_1 &\leq M|\beta|S_1 + \left|\frac{2\beta}{(E'_n-E'_0)}\right|S_3 + M\beta^2 t\, S_5,\\
I_2 &\leq M|\beta|S_2 + \left|\frac{2\beta}{(E'_n-E'_0)}\right|S_4 + M\beta^2 t\, S_6,
\end{aligned}
\tag{3.14}
$$

where S_i are small divisor sums, defined bellow (3.17) .

Proof The bounds on those integrals are

$$
\begin{aligned}
|I_1| &\leq \left|-\beta\sum_{m_1}\frac{V_n^{m_100}c_{m_1}^{(r)*}e^{i(E'_n+E'_{m_1}-2E'_0)t}}{(E'_n+E'_{m_1}-2E'_0)}\right| + \left|\beta\frac{e^{i(E'_n-E'_0)t}-1}{(E'_n-E'_0)}\right|\left|\sum_{m_1}\frac{V_n^{m_100}V_{m_1}^{000}}{(E'_n+E'_{m_1}-2E'_0)}\right|\\
&\quad + \beta^2\left|\int_0^t ds\sum_{m_1,m_2}\frac{V_n^{m_100}V_{m_1}^{m_200}c_{m_2}^{(r)}}{(E'_n+E'_{m_1}-2E'_0)}e^{i(E'_n-E'_{m_2})s}\right|\\
&\quad + \beta^2\left|\int_0^t ds\sum_{m_1}\sum_{m_2\neq m_1}\frac{2V_n^{m_100}V_{m_1}^{m_200}c_{m_2}^{(r)*}}{(E'_n+E'_{m_1}-2E'_0)}e^{i(E'_n+E'_{m_2}-2E'_0)s}\right|\\
&\leq |\beta|\sum_{m_1}\left|\frac{V_n^{m_100}c_{m_1}^{(r)*}}{(E'_n+E'_{m_1}-2E'_0)}\right| + \left|\frac{2\beta}{(E'_n-E'_0)}\right|\left|\sum_{m_1}\frac{V_n^{m_100}V_{m_1}^{000}}{(E'_n+E'_{m_1}-2E'_0)}\right|\\
&\quad + \beta^2 t\left(\sum_{m_1,m_2}\left|\frac{V_n^{m_100}V_{m_1}^{m_200}c_{m_2}^{(r)}}{(E'_n+E'_{m_1}-2E'_0)}\right| + \sum_{m_1}\sum_{m_2\neq m_1}\left|\frac{2V_n^{m_100}V_{m_1}^{m_200}c_{m_2}^{(r)*}}{(E'_n+E'_{m_1}-2E'_0)}\right|\right)\\
&\leq M|\beta|\sum_{m_1}\left|\frac{V_n^{m_100}e^{-a|x_{m_1}|}}{(E'_n+E'_{m_1}-2E'_0)}\right| + \left|\frac{2\beta}{(E'_n-E'_0)}\right|\left|\sum_{m_1}\frac{V_n^{m_100}V_{m_1}^{000}}{(E'_n+E'_{m_1}-2E'_0)}\right|
\end{aligned}
$$

$$+ M\beta^2 t\left(\sum_{m_1,m_2}\left|\frac{V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|x_{m_2}|}}{(E_n'+E_{m_1}'-2E_0')}\right|+\sum_{m_1}\sum_{m_2\neq m_1}\left|\frac{2V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|x_{m_2}|}}{(E_n'+E_{m_1}'-2E_0')}\right|\right)$$

$$= M|\beta|S_1+\left|\frac{2\beta}{(E_n'-E_0')}\right|S_3+M\beta^2 t\, S_5. \tag{3.15}$$

Where at the last step we have used the hypotheses that

$$|c_n^{(r)}|\leq Me^{-a|x_n|}, \tag{3.16}$$

for some $a>0$, $M<\infty$. This hypothesis will be verified self-consistently latter by a bootstrap argument, and it will be shown that one can take, $a=\gamma-\varepsilon$, where ε is arbitrarily small. Additionally, to simplify the calculation we have defined

$$\begin{aligned}
S_1 &= \sum_{m_1}\left|\frac{V_n^{m_1 00}e^{-a|x_{m_1}|}}{(E_n'+E_{m_1}'-2E_0')}\right|, \qquad S_2=\sum_{m_1\neq n}\left|\frac{2V_n^{m_1 00}e^{-a|x_{m_1}|}}{(E_n'-E_{m_1}')}\right|,\\
S_3 &= \left|\sum_{m_1}\frac{V_n^{m_1 00}V_{m_1}^{000}}{(E_n'+E_{m_1}'-2E_0')}\right|, \qquad S_4=\left|\sum_{m_1\neq n}\frac{2V_n^{m_1 00}V_{m_1}^{000}}{(E_n'-E_{m_1}')}\right|,\\
S_5 &= \sum_{m_1,m_2}\left|\frac{V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|x_{m_2}|}}{(E_n'+E_{m_1}'-2E_0')}\right|+\sum_{m_1}\sum_{m_2\neq m_1}\left|\frac{2V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|x_{m_2}|}}{(E_n'+E_{m_1}'-2E_0')}\right|,\\
S_6 &= \sum_{m_1\neq n,m_2\neq m_1}\left|\frac{4V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|x_{m_2}|}}{(E_n'-E_{m_1}')}\right|+\sum_{m_1\neq n}\sum_{m_2}\left|\frac{2V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|x_{m_2}|}}{(E_n'-E_{m_1}')}\right|.
\end{aligned} \tag{3.17}$$

Similarly for I_2

$$|I_2|\leq M|\beta|\sum_{m_1\neq n}\left|\frac{2V_n^{m_1 00}e^{-a|m_1|}}{(E_n'-E_{m_1}')}\right|+\left|\frac{2\beta}{(E_n'-E_0')}\right|\left|\sum_{m_1\neq n}\frac{2V_n^{m_1 00}V_{m_1}^{000}}{(E_n'-E_{m_1}')}\right|$$

$$+ M\beta^2 t\left(\sum_{m_1\neq n,m_2\neq m_1}\left|\frac{4V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|m_2|}}{(E_n'-E_{m_1}')}\right|+\sum_{m_1\neq n}\sum_{m_2}\left|\frac{2V_n^{m_1 00}V_{m_1}^{m_2 00}e^{-a|m_2|}}{(E_n'-E_{m_1}')}\right|\right)$$

$$= M|\beta|S_2+\left|\frac{2\beta}{(E_n'-E_0')}\right|S_4+M\beta^2 t\, S_6. \tag{3.18}$$

□

We would like to prove that $I_{1,2}$ are uniformly bounded for t satisfying $\beta^2 t<O(1)$. For this purpose, as will be shown below, it will be sufficient to bound only $S_{1,2}$. It will enable to bound all other sums.

In order to use Aizenman-Molchanov (A-M) type of estimate [46], we rewrite the sums (3.17) using integrals over the Green's function. The key observation is (3.19). The Green's function could be written as

$$G_E(x,y)=\sum_l\frac{\phi_l(x)\phi_l(y)}{E-E_l'}.$$

Multiplication by $\phi_n(x)\,\phi_0^2(x)\,\phi_m(y)$ and integrating produces

$$\int dxdy\phi_n(x)\,\phi_0^2(x)\,G_E(x,y)\,\phi_m(y) = \sum_l \frac{[\int dx\phi_n(x)\,\phi_0^2(x)\,\phi_l(x)][\int dy\phi_l(y)\,\phi_m(y)]}{E-E_l'}$$
$$= \frac{\int dx\phi_n(x)\,\phi_0^2(x)\,\phi_m(x)}{E-E_m'} = \frac{V_n^{m00}}{E-E_m'}. \tag{3.19}$$

We outline the estimate of a bound for the Green's function that will be useful in order to produce a bound on sums of the form (3.17). The detailed derivation will be presented elsewhere [47].

From the Furstenberg theorem [48] combined with [38, 39], using the continuity of the Lyapunov exponent as a function of energy [49] one finds that with probability $1-\varepsilon$ all the eigenfunctions fall of exponentially

$$\phi_E \sim e^{-\gamma(E)|n|} \quad \text{for } n > \bar{n} \tag{3.20}$$

where ε is a decreasing function of $\bar{n}$. For each $\bar{n}$ we have excluded bad realizations where (3.20) does not hold.

Consider a combination of eigenenergies

$$f = \sum_k c_k E_k.$$

The Feynman-Hellman theorem implies

$$\frac{\partial f}{\partial \varepsilon_j} = \sum_k c_k\,|\phi_i(j)|^2. \tag{3.21}$$

For $j > \bar{n}$ this derivative is dominated by one term of the order of $\phi_{E_i}(j) \sim e^{-\gamma(E_i)|j|}$ as one can see from (3.20). The average $\langle |f|^{-s}\rangle_\varepsilon$ where the bad realizations are excluded is bounded. In order to show this, the integral involved in the calculations of the average is transformed to variables so that f is one of them. The Jacobian is estimated with the help of (3.21).

The A-M estimate [46] for the Green's function of the Anderson model is

$$\left\langle |G_E(x,y)|^s \right\rangle \le F\exp(-c\,|x-y|) \quad \text{with } F, c > 0 \tag{3.22}$$

where $0 < s < 1$. We use this estimate in the present paper by replacing the energy E by a combination of eigenenergies. Assuming localization one can show that the bound is satisfied also for this case if the average is over the set of realizations of measure $1-\varepsilon$, where a specific set of "badly" behaved realizations, defined above (3.20), was eliminated. The Green's function satisfies

$$|G_E(x,y)|^s \le \sum_n \frac{|\phi_n(x)|^s\,|\phi_n(y)|^s}{|E-E_n|^s}.$$

Using the Hölder inequality for each term in the sum one finds

$$\left\langle |G_E(x,y)|^s \right\rangle \le \sum_n \left\langle |\phi_n(x)|^{ps}\,|\phi_n(y)|^{ps} \right\rangle^{1/p} \left\langle \frac{1}{|E-E_n|^{sq}} \right\rangle^{1/q}, \tag{3.23}$$

for $q, p > 1$ and $1/q + 1/p = 1$. If $(E - E_n) = \sum_k c_k E_k$ the term

$$\left\langle \frac{1}{|E - E_n|^{sq}} \right\rangle_\varepsilon \leq D_\varepsilon \tag{3.24}$$

is bounded if $0 < sq < 1$ and $\langle . \rangle_\varepsilon$ is an average which does not include the bad realizations of the potentials. Because of exponential localization [48–51]

$$\sum_n \left\langle |\phi_n(x)|^{ps} |\phi_n(y)|^{ps} \right\rangle^{1/p} \leq F'_{p,\varepsilon} e^{-c|x-y|}, \tag{3.25}$$

with $0 < c < s \cdot \min_E \gamma(E)$. The parameter q can be made arbitrarily close to 1, therefore in our case (3.22) holds for $0 < s < 1 - \eta$ with η arbitrarily small. If the energies E_j are replaced by the E'_j which are obtained from E_j by removing the secular terms (2.33), then (3.22) also holds for sufficiently small β. Hence the Green's function with E replaced by some combination of the energies E'_j, that will be denoted by $G_E^{(1)}$ satisfies the A-M bound

$$\left\langle \left| G_E^{(1)}(x, y) \right|^s \right\rangle_\varepsilon \leq F_\varepsilon \exp(-c|x - y|), \quad F_\varepsilon, c > 0, \tag{3.26}$$

for $0 < s < 1 - \eta$ with η arbitrarily small.

The energies E_n are indexed by their centers of localization (see discussion that follows (2.4)). When the parameters of the model, such as the diagonal energies, vary the energies change continuously. The effects of such variations were taken into account in the above discussion. The results of the above discussion can be generalized to a situation where there is a finite number of discontinuities of the energies as a function of the potentials. This is done by dividing the integration domain into regions where continuity holds. Unfortunately, we cannot assure that these discontinuities are not found on arbitrarily small energy scales for all realizations of the random potential. There may be realizations for which "double-humped" states are found with centers of localization that are typically very far from each other. For such situations avoiding-crossings occur and produce a discontinuity in the energy with a gap which is exponentially small in the distance between the "humps". The measure of such realizations, ε, decreases with the distance between the "humps" (see (2.4)). In this work we consider only realizations for which the energy is a piecewise continuous function of the potentials on some scale, that is determined by ε, and (3.26) holds.

Proposition 3.2 *For all* $\eta > 0$, $0 < b < \gamma$, $0 < s < 1$,

$$\Pr\left(\left| \frac{V_n^{m00}}{E - E_m} \right| \geq C_0 e^{-(\gamma-\eta)|x_n|} e^{-b|x_m|} \right) \leq K_\varepsilon e^{-\eta s |x_n|}. \tag{3.27}$$

Proof The RHS of (3.19) can be bounded by

$$\begin{aligned} &\left| \int dxdy \phi_n(x) \phi_0^2(x) G_E^{(1)}(x, y) \phi_m(y) \right| \\ &\leq \int dxdy\, |\phi_n(x)| \left| \phi_0^2(x) \right| \left| G_E^{(1)}(x, y) \right| |\phi_m(y)| \\ &\leq D_{\omega,\varepsilon}^4 e^{\varepsilon(|x_n|+|x_m|)} \int dxdy \cdot e^{-\gamma(|x-x_n|+|y-x_m|+2|x|)} \left| G_E^{(1)}(x, y) \right|, \end{aligned} \tag{3.28}$$

where we have used the bound (2.3), for a.e. (ω, E). We rewrite

$$|x-x_n|+|y-x_m|+2|x| = (|x-x_n|+|x|)+\frac{b}{\gamma}(|y-x_m|+|x-y|+|x|) + \left(1-\frac{b}{\gamma}\right)(|y-x_m|+|x|)-\frac{b}{\gamma}|x-y|, \tag{3.29}$$

and using the triangle inequality on the RHS for $0 < b < \gamma$,

$$|x-x_n|+|y-x_m|+2|x| \geq |x_n|+\frac{b}{\gamma}|x_m|-\frac{b}{\gamma}|x-y|+\left(1-\frac{b}{\gamma}\right)|x|. \tag{3.30}$$

This yields

$$\left|\int dxdy\phi_n(x)\phi_0^2(x)G_E^{(1)}(x,y)\phi_m(y)\right| \leq D_{\omega,\varepsilon}^4 e^{-(\gamma-\varepsilon)|x_n|}e^{-(b-\varepsilon)|x_m|}\int dxdy\cdot e^{-(\gamma-b)|x|}e^{b|x-y|}|G_E^{(1)}(x,y)|. \tag{3.31}$$

We define

$$I_{AM} = D_{\omega,\varepsilon}^4 \int dxdy\cdot e^{-(\gamma-b)|x|}e^{b|x-y|}|G_E^{(1)}(x,y)|, \tag{3.32}$$

which satisfies

$$\begin{aligned}\langle|I_{AM}|^s\rangle &\leq \left\langle D_{\omega,\varepsilon}^{4s}\left(\int dxdy\cdot e^{-(\gamma-b)|x|}e^{b|x-y|}|G_E^{(1)}(x,y)|\right)^s\right\rangle \\ &\leq \int \langle D_{\omega,\varepsilon}^{4s}|G_E^{(1)}(x,y)|^s\rangle e^{-(\gamma-b)s|x|}e^{bs|x-y|}dydx \\ &\leq (F\langle D_{\omega,\varepsilon}^{8s}\rangle)^{1/2}\int\langle|G_E^{(1)}(x,y)|^{2s}\rangle^{1/2}e^{-(\gamma-b)s|x|}e^{bs|x-y|}dydx \\ &\leq (F\langle D_{\omega,\varepsilon}^{8s}\rangle)^{1/2}\int e^{-c|x-y|}e^{-(\gamma-b)s|x|}e^{bs|x-y|}dydx = K_\varepsilon < \infty,\end{aligned} \tag{3.33}$$

where (3.22) was used in the last step and it was assumed that $s < \frac{1}{2}$. The number b is in the interval $0 < b < \min(\gamma, \frac{c}{2s})$. For the realizations we study, (2.4) holds and therefore $\langle D_{\omega,\varepsilon}^{8s}\rangle, K_\varepsilon < \infty$.That implies

$$\Pr(I_{AM} \geq e^{\eta|x_n|}) \leq Ke^{-\eta s|x_n|}, \tag{3.34}$$

where K_ε is a constant, which was defined in (3.33) and $\eta > 0$. Using the relation (3.19) and (3.31) we obtain

$$\Pr\left(\left|\frac{V_n^{m00}}{E-E_m}\right| \geq C_0 e^{-(\gamma-\eta)|x_n|}e^{-b|x_m|}\right) \leq K_\varepsilon e^{-\eta s|x_n|}. \tag{3.35}$$

□

Therefore the probabilistic bound on an element of S_1 is

$$\Pr\left(\left|\frac{V_n^{m_1 00} e^{-a|x_{m_1}|}}{(E'_n + E'_{m_1} - 2E'_0)}\right| \geq C_0 e^{-(\gamma-\eta)|x_n|} e^{-(a+b)|x_{m_1}|}\right) \leq K_\varepsilon e^{-\eta s |x_n|}, \tag{3.36}$$

which means that

$$\Pr(S_1 > D_1 e^{-(\gamma-\eta)|x_n|}) \leq K_\varepsilon e^{-\eta s |x_n|}. \tag{3.37}$$

Similarly for S_2,

$$\Pr(S_2 > D_2 e^{-(\gamma-\eta)|x_n|}) \leq K_\varepsilon e^{-\eta s |x_n|}. \tag{3.38}$$

Using the fact that

$$\begin{aligned} |V_n^{m00}| &\leq D^4_{\omega,\varepsilon} e^{\varepsilon(|x_n|+|x_m|)} \int dx e^{-\gamma(|x-x_n|+|x-x_m|+2|x|)} \\ &\leq D^4_{\omega,\varepsilon} e^{-(\gamma-\varepsilon-\varepsilon')(|x_n|+|x_m|)} \int dx e^{-\varepsilon'(|x-x_n|+|x-x_m|+2|x|)} \\ &\leq V_0 e^{-(\gamma-\varepsilon-\varepsilon')(|x_n|+|x_m|)} \end{aligned} \tag{3.39}$$

we can bound the second term in (3.15) by

$$\begin{aligned} \left|\frac{2\beta}{(E'_n - E'_0)}\right| S_3 &= \left|\frac{2\beta}{(E'_n - E'_0)}\right| \left|\sum_{m_1} \frac{V_n^{m_1 00} V_{m_1}^{000}}{(E'_n + E'_{m_1} - 2E'_0)}\right| \\ &\leq V_0 \left|\frac{2\beta e^{-(\gamma-\varepsilon-\varepsilon')|x_n|}}{(E'_n - E'_0)}\right| \sum_{m_1} \left|\frac{V_{m_1}^{000} e^{-(\gamma-\varepsilon-\varepsilon')|x_{m_1}|}}{(E'_n + E'_{m_1} - 2E'_0)}\right|. \end{aligned} \tag{3.40}$$

In order to bound the prefactor before the above sum we write

$$\frac{1}{(E'_n - E'_0)} = \int dx dy\, \phi_n(x) G^{(1)}_{E'_0}(x, y) \phi_n(y) \equiv I_0, \tag{3.41}$$

and now using A-M estimates

$$\langle |I_0|^s \rangle \leq D^2 \int dx dy\, e^{-\gamma s (|x-x_n|+|y-x_n|)} \langle |G^{(1)}_{E'_0}(x, y)|^s \rangle = K'_\varepsilon < \infty. \tag{3.42}$$

Therefore using (3.22)

$$\Pr\left(I_0 e^{-\gamma|x_n|} = \frac{e^{-\gamma|x_n|}}{(E'_n - E'_0)} \geq e^{-(\gamma-\eta')|x_n|}\right) \leq K'_\varepsilon e^{-\eta' s |x_n|}. \tag{3.43}$$

Using the results of (3.43) and (3.37) and the fact, that if $a_1 a_2 > b_1 b_2$ at least one of the a_i is larger than one of the b_i, we get

$$\Pr\left(\left|\frac{2\beta}{(E'_n - E'_0)}\right| S_3 \geq \beta D_3 e^{-(\gamma-\bar{\eta})|x_n|}\right) \leq 2\bar{K}_\varepsilon e^{-\bar{\eta} s |x_n|}, \tag{3.44}$$

where $\bar{K}_\varepsilon = \max(K_\varepsilon, K'_\varepsilon)$ and $\bar{\eta} = \min(\eta, \eta')$ and D_3 is a constant. Similarly for the second term in (3.18)

$$\Pr\left(\left|\frac{2\beta}{(E'_n - E'_0)}\right| S_4 \geq \beta D_4 e^{-(\gamma-\bar{\eta})|x_n|}\right) \leq 2\bar{K}_\varepsilon e^{-\bar{\eta} s |x_n|}. \tag{3.45}$$

The sums $S_{5,6}$ are bounded by

$$\begin{aligned} S_5 &\leq \sum_{m_1} \left| \frac{V_n^{m_1 00} F e^{-(\gamma-\varepsilon)|x_{m_1}|}}{(E'_n + E'_{m_1} - 2E'_0)} \right| + \sum_{m_1} \left| \frac{2V_n^{m_1 00} F e^{-(\gamma-\varepsilon)|x_{m_1}|}}{(E'_n + E'_{m_1} - 2E'_0)} \right|, \\ S_6 &\leq \sum_{m_1 \neq n} \left| \frac{4V_n^{m_1 00} F e^{-(\gamma-\varepsilon)|x_{m_1}|}}{(E'_n - E'_{m_1})} \right| + \sum_{m_1 \neq n} \left| \frac{2V_n^{m_1 00} F e^{-(\gamma-\varepsilon)|x_{m_1}|}}{(E'_n - E'_{m_1})} \right| \end{aligned} \tag{3.46}$$

since, using (3.39)

$$\sum_{m_2} |V_{m_1}^{m_2 00}| e^{-a|x_{m_2}|} \leq V_0 e^{-(\gamma-\varepsilon)|x_{m_1}|} \sum_{m_2} e^{-(\frac{\gamma}{2}+a)|x_{m_2}|} \leq F e^{-(\gamma-\varepsilon)|x_{m_1}|}.$$

Therefore using the same considerations which were used to bound $S_{1,2}$ we get

$$\Pr(S_{5,6} \geq D_{5,6} e^{-(\gamma-\eta)|x_n|}) \leq K_\varepsilon e^{-\eta s |x_n|}. \tag{3.47}$$

The bound on each of the sums satisfies

$$\Pr(S_i \geq D_i e^{-(\gamma-\eta_i)|x_n|}) \leq K_{i,\varepsilon} e^{-\eta_i s |x_n|} \quad \text{with } i \subseteq \{1, \dots, 6\}. \tag{3.48}$$

Finally,

$$\Pr(|I_1 + I_2| e^{(\gamma-\bar{\eta})|x_n|} \geq C_1 M |\beta| + C_2 |\beta| + C_3 M \beta^2 t) \leq 6\bar{K}_\varepsilon e^{-\bar{\eta} s |x_n|}$$

and therefore the solution of the linear part

$$c_n^{(r)} = -\frac{V_n^{000}}{E'_n - E'_0} (e^{i(E'_n - E'_0)t} - 1) + I_1 + I_2 \tag{3.49}$$

satisfies:

Proposition 3.3

$$\Pr(|c_n^{(r)}| e^{(\gamma-\bar{\eta})|x_n|} \geq C_0 + (C_1 M + C_2)|\beta| + C_3 M \beta^2 t) \leq 7\bar{K}_\varepsilon e^{-\bar{\eta} s |x_n|}, \tag{3.50}$$

where C_0 is defined in (3.35) *and* $0 < \bar{\eta} < \gamma$.

3.2 The Bootstrap Argument

We now complete the estimates by controlling the nonlinear terms. Equation (3.4) can be integrated for $n \neq 0$ to take the form of

$$c_n^{(r)} = J_1 + J_2 + J_3 + \Delta J_1, \tag{3.51}$$

where J_1 is the integral of the linear part, namely, the RHS of (3.5), while J_2 and J_3 are the integrals of the terms of order β^2 and β^3, respectively. Substitution of the terms of order β and β^2 of (3.4) would add to I_1 and I_2 a contribution, ΔJ_1, which will be estimated latter (see (3.61)).

Proposition 3.4

$$\begin{aligned} |J_2| &\le 3F_2 M^2 \beta^2 t e^{-(\gamma-\varepsilon)|x_n|}, \\ |J_3| &\le F_3 M^3 \beta^3 t e^{-(\gamma-\varepsilon)|x_n|}. \end{aligned} \tag{3.52}$$

Proof Using the hypothesis (3.16) one finds

$$|J_2| \le 3\beta^2 t \sum_{m_1,m_2} |V_n^{m_1 m_2 0}||c_{m_1}||c_{m_2}| \le 3M^2\beta^2 t \sum_{m_1,m_2} |V_n^{m_1 m_2 0}| e^{-a(|x_{m_1}|+|x_{m_2}|)} \tag{3.53}$$

and

$$|J_3| \le \beta^3 t \sum_{m_1,m_2,m_3} |V_n^{m_1 m_2 m_3}||c_{m_1}^{(r)*} c_{m_2}^{(r)} c_{m_3}^{(r)}| \le M^3 \beta^3 t \sum_{m_1,m_2,m_3} |V_n^{m_1 m_2 m_3}| e^{-a(|x_{m_1}|+|x_{m_2}|+|x_{m_3}|)}. \tag{3.54}$$

To bound the sums we use (2.3),

$$\begin{aligned} \Delta\tilde{c}_n^{(2)} &= \sum_{m_1,m_2} |V_n^{m_1 m_2 0}| e^{-a(|x_{m_1}|+|x_{m_2}|)} \\ &\le D_{\omega,\varepsilon}^4 \int dx \sum_{m_1,m_2} e^{\varepsilon(|x_n|+|x_{m_1}|+|x_{m_2}|)} e^{-\gamma(|x|+|x-x_n|+|x-x_{m_1}|+|x-x_{m_2}|)} e^{-a(|x_{m_1}|+|x_{m_2}|)} \\ &= D_{\omega,\varepsilon}^4 e^{\varepsilon|x_n|} \int dx\ e^{-\gamma(|x-x_n|+|x|)} \left(\sum_m e^{-\gamma|x-x_m|} e^{-(a-\varepsilon)|x_m|} \right)^2 \end{aligned} \tag{3.55}$$

and since $\gamma > (a-\varepsilon)$,

$$\sum_m e^{-\gamma|x-x_m|} e^{-(a-\varepsilon)|x_m|} \le \bar{F} e^{-(a-\varepsilon)|x|}, \tag{3.56}$$

which gives

$$\begin{aligned} \Delta\tilde{c}_n^{(2)} &\le \bar{F} D_{\omega,\varepsilon}^4 e^{-(\gamma-\varepsilon)|x_n|} \int dx\ e^{-2(a-\varepsilon)|x|} \\ &= F_2(\omega,\gamma,a,\varepsilon) e^{-(\gamma-\varepsilon)|x_n|}. \end{aligned} \tag{3.57}$$

For the other sum,

$$\begin{aligned} \Delta\tilde{c}_n^{(3)} &= \sum_{m_1,m_2,m_3} |V_n^{m_1 m_2 m_3}| e^{-a(|x_{m_1}|+|x_{m_2}|+|x_{m_3}|)} \\ &\le D_{\omega,\varepsilon}^4 e^{\varepsilon|x_n|} \int dx\ e^{-\gamma|x-x_n|} \left(\sum_m e^{-\gamma|x-x_m|} e^{-(a-\varepsilon)|x_m|} \right)^3 \end{aligned} \tag{3.58}$$

taking $\gamma < 3(a-\varepsilon)$,

$$\begin{aligned}\Delta\tilde{c}_n^{(3)} &\le F'(\omega,\gamma,a,\varepsilon)e^{\varepsilon|x_n|}\int dx\ e^{-\gamma|x-x_n|}e^{-3(a-\varepsilon)|x|}\\ &= F'(\omega,\gamma,a,\varepsilon)e^{\varepsilon|x_n|}\int dx\ e^{-\gamma(|x-x_n|+|x|)}e^{-(3(a-\varepsilon)-\gamma)|x|}\\ &\le F_3(\omega,\gamma,a,\varepsilon)e^{-(\gamma-\varepsilon)|x_n|}. \end{aligned} \tag{3.59}$$

The resulting bounds prove the proposition. □

From (3.8) and (3.12) it is clear that

$$\begin{aligned}|\Delta J_1| &\le \beta t\sum_{m_1}\left|\frac{V_n^{m_1 00}}{(E'_n+E'_{m_1}-2E'_0)}\right|(\beta^2M^2\Delta\tilde{c}_{m_1}^{(2)}+\beta^3M^3\Delta\tilde{c}_{m_1}^{(3)})\\ &\quad+\beta t\sum_{m_1\ne n}\left|\frac{2V_n^{m_1 00}}{(E'_n-E'_{m_1})}\right|(\beta^2M^2\Delta\tilde{c}_{m_1}^{(2)}+\beta^3M^3\Delta\tilde{c}_{m_1}^{(3)}). \end{aligned} \tag{3.60}$$

Using the inequalities (3.57) and (3.59) we obtain

$$\begin{aligned}|\Delta J_1| &\le \beta t\sum_{m_1}\left|\frac{V_n^{m_1 00}}{(E'_n+E'_{m_1}-2E'_0)}\right|(\beta^2M^2F_2e^{-(\gamma-\varepsilon)|x_{m_1}|}+\beta^3M^3F_3e^{-(\gamma-\varepsilon)|x_{m_1}|})\\ &\quad+\beta t\sum_{m_1\ne n}\left|\frac{2V_n^{m_1 00}}{(E'_n-E'_{m_1})}\right|(\beta^2M^2F_2e^{-(\gamma-\varepsilon)|x_{m_1}|}+\beta^3M^3F_3e^{-(\gamma-\varepsilon)|x_{m_1}|})\\ &\le M^2\beta^3t(F_2+\beta MF_3)(S_1+S_2). \end{aligned} \tag{3.61}$$

Since $S_{1,2}$ are bounded $|\Delta J_1|$ is also bounded with the same probability.
The above results give:

Proposition 3.5 *With probability of the order* $1-Ce^{-\bar{\eta}s|x_n|}$ *and for realizations, where* (2.4) *is satisfied*:

$$M=\sup_n c_n^{(r)}e^{a|x_n|}\le C_0+C_2|\beta|+|\beta|C_1M+C_3M\beta^2t+F_2M^2\beta^2t\ +F_3M^3\beta^3t.$$

By subtracting $|\beta|C_1M$ from both sides of the inequality and dividing by $(1-|\beta|C_1)$ we obtain

$$M\le\frac{1}{1-|\beta|C_1}[C_0+C_2|\beta|+C_3M\beta^2t+(F_2M^2\beta^2t\ +F_3M^3\beta^3t)], \tag{3.62}$$

and renaming the constants produces

$$M\le G_0+G_1M\beta^2t+G_2M^2\beta^2t+G_3M^3\beta^3t, \tag{3.63}$$

where G_i are constants.

Setting $M(t=0)=G_0$ we check if $M(t)$ can reach $2G_0$, i.e., whether there is a time, t_0, such that $M(t_0)=2G_0$. Inserting into the inequality we have

$$2G_0 \leq G_0 + G_1\beta^2 t_0(2G_0) + (G_2(2G_0)^2\beta^2 t_0 + G_3(2G_0)^3\beta^3 t_0). \tag{3.64}$$

For, $t \leq t_0$,

$$\beta^2 t \cdot (2G_1 + 4G_2G_0 + 8\beta G_3 G_0^2) \leq 1 \tag{3.65}$$

holds, if $\beta^2 t_0$ is small enough, therefore (3.64) cannot be satisfied, hence, by continuity, $M(t) < 2G_0$. This proves the hypothesis (3.16) with $a = \gamma - \bar{\eta}$, taking $\bar{\eta} > \varepsilon$ and $\varepsilon, \bar{\eta}$ arbitrarily small, for $\beta^2 t < O(1)$.

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